Eutrigon Theorem: a new* 'twin' to the theorem of Pythagoras

Geometric Form

The area of any eutrigon (shaded in fig. 69) is equal to the sum of the areas of the equilateral triangles on its legs *a* and *b*, minus the area of the equilateral triangle on its hypotenuse, *c*.

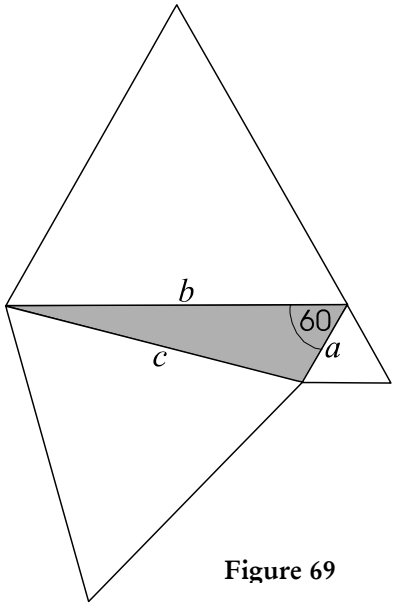

Figure 69

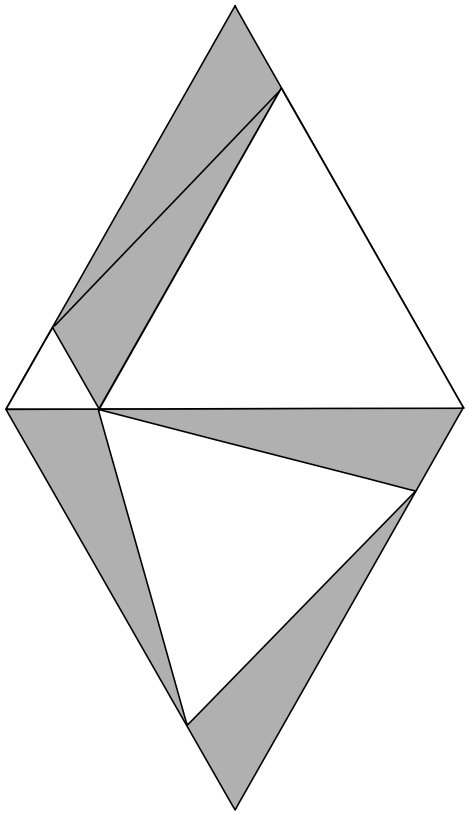

Figure 70
A 'look-&-see' proof. Note: All triangles which *appear* equilateral *are* equilateral. See next page for an expanded exposition.

* The closest theorem I found to this is the Napoleon Theorem (Appendix 2)

Detailing the proof

From Fig. 71 it can be seen that:
1) Triangles marked "Q" are *congruent eutrigons* since each contains one 60^E angle, and their respective sides are equal.
2) Triangles A, B, and C are equilateral
3) A and B are the equilateral triangles on the legs of eutrigon Q, and C is the equilateral triangle on its hypotenuse.
4) The overall diamond shape is a rhombus consisting of an upper and lower equilateral triangle of *identical area*.
5) Point "4" means that, expressed in terms of areas,
A+B+2Q = C + 3Q
Thus, Q = A + B - C
ˆthe ***Eutrigon Theorem*** is proved

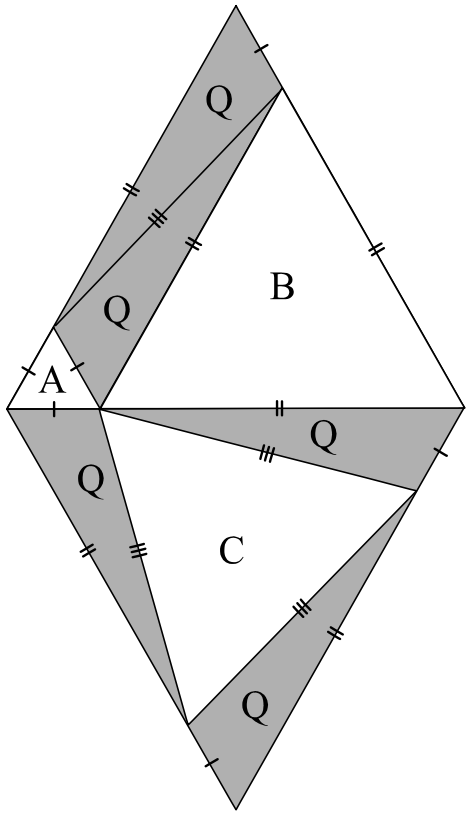

Figure 71

Now, some readers may not be satisfied that in the 'look & see' proof given above, that eutrigon Q may be *any* eutrigon. Before we prove this more rigorously let us *imagine* whether it is so.

Imagine an *animation* of equilateral-triangle C rotating while maintaining its equilateral shape but varying in size such that its vertices 'slide along' the outer equilateral triangle it is touching (see fig. 72), ...then you may see in your 'mind's eye' the fact that every possible shape of eutrigon *is* accounted for in the constructions of figures 70 and 71. But let's take a closer look from an additional perspective. In so doing, we will once again apply the principle of *relative units*.

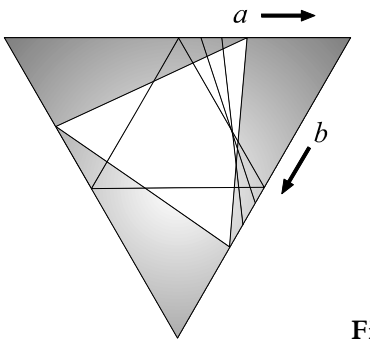

Figure 72 Showing how the encompassing large equilateral triangle can accommodate every possible eutrigon shape

Can "Q" be *any* eutrigon?

The visual proof as presented in figures 70 and 71 includes no stipulation as to the *absolute size* of the figure and its component shapes. The geometric relationships hold whether the diagram fills a football field, or whether it fits on the back of an envelope with room to spare. The entire diagram is scaleable: it can be made larger or smaller without affecting the geometrical relations within it. Thus the particular eutrigon in the diagram can be any size at all, and this does not alter the internal relations of the diagram, and therefore our theorem is unaffected by scaling the entire diagram up or down by some factor.

But we need to still check whether or not the geometric construction also accounts for every possible *shape* of eutrigon. This is a little trickier.

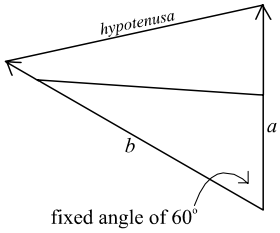

Figure 73

Our definition of a *eutrigon* is of a triangle with one angle equal to 60-degrees. Its shape is thus solely determined by changing the *relative lengths* of its legs *a* and *b* (the sides adjacent to the 60-degree angle). From this point of view, the hypotenuse length is passively determined to the extent that it merely joins the 'free' end of leg *a* with that of leg *b, and this length is determined by* the fact that the angle between legs *a* and *b is* fixed at 60-degrees. So the length of leg *a* relative *to* the length of leg *b* is all the information we need to determine the shape of *any* eutrigon (fig. 73). This makes our analysis of possible shapes so much easier.

We may combine the two values *a* and *b* into a single value reflecting their *relative* lengths—the ratio, *a/b*. This ratio thus defines the shape of any eutrigon. If we make *a* vanishingly small, *b* (in comparison) 'becomes' immensely large; *if the ratio a/b is the same, the shape is the same.* The following three possibilities exist: $a = b$; $a > b$; or, $a < b$. When a = b, the eutrigon is the special case of an equilateral triangle. In every other case, one side is smaller than the other. Let us define *a* to be always the *smaller* of the two legs *a* and *b*. Hence, *a/b* #1 for any eutrigon. So if our geometric construction can accommodate eutrigons *of every leg-ratio (a/b) between zero and one*, then the proof holds for *all* eutrigons.

We are now ready to refer back to our visual proof of fig. 70. The lower half of that diagram has been reproduced in fig. 72, except this time, the dynamics of eutrigon-change is hinted at.

We first stipulate that the three corners of the large equilateral triangle always contain three congruent eutrigons (shaded in the figure), whose shape may vary but whose *leg-length sum (a + b) is equal to the side-length of the outer equilateral triangle*. This latter point means that they will always fit around the edge of the equilateral triangle in 'tip-to-toe' fashion, as in fig. 72.